AD-A090 539 MARYLAND UNIV COLLEGE PARK DEPT OF COMPUTER SCIENCE F/G 5/8
CONVEX DIGITAL SOLIDS.(U)
AUG 80 C E KIM, A ROSENFELD AFOSR-77-3271
UNCLASSIFIED TR-929 AFOSR-TR-80-1036 NL

1 OF 1
AD-
A090539

END
DATE
FILMED
11 80
DTIC

AFOSR-TR- 80-1036

LEVEL III

AD A090539

DTIC ELECTE OCT 17 1980

UNIVERSITY OF MARYLAND

UNIVERSITY OF MARYLAND
COMPUTER SCIENCE CENTER

COLLEGE PARK, MARYLAND
20742

DDC FILE COPY

80 10 9 091

TR-929
AFOSR-77-3271

August, 1980

CONVEX DIGITAL SOLIDS

Chul E. Kim
Department of Computer Science
University of Maryland
College Park, MD 20742

Azriel Rosenfeld
Computer Vision Laboratory
Computer Science Center
University of Maryland
College Park, MD 20742

ABSTRACT

A definition of convexity of digital solids is introduced. Then it is proved that a digital solid is convex if and only if it has the chordal triangle property. Other geometric properties which characterize convex digital regions are shown to be only necessary, but not sufficient, conditions for a digital solid to be convex. An efficient algorithm is presented that determines whether or not a digital solid is convex.

The support of the U.S. Air Force Office of Scientific Research under Grant AFOSR-77-3271 is gratefully acknowledged, as is the help of Kathryn Riley in preparing this paper.

1. Introduction

Discrete geometry has applications in image processing, pattern recognition and other areas. Before an image of an object is processed by a computer, it is digitized to yield a finite subset of digital points called a digital image. Then, various types of operations such as template matching and geometric property measurement are performed on sets of digital points.

Convexity is an important and useful geometric property. Since digital image processing has been restricted almost entirely to 2-dimensional images, digital convexity has been studied only for digital regions, that is, 8-connected [10] finite subsets of digital points in the plane. Sklansky in [13] defined a digital region to be convex if and only if there is a convex region whose image (under digitization) is the digital region. He then showed that a digital region is convex if and only if its minimum-perimeter polygon is convex [13,14]. A digital region R is said to have the line property if every digital point on the line segment between any two points of R is a point of R. In [7] Minsky and Papert defined a digital region to be convex if and only if it has the line property. However, no useful results regarding convex digital regions have been derived from this definition. Let d_1, d_2 be two points of R and R' the union of all cells (grid squares) whose centers are points of R. Then $P(R;d_1,d_2)$ denotes the area bounded by the

line segment $\overline{d_1 d_2}$ and the boundary of R'. R is said to have the area property if for any two points d_1, d_2 of R, every digital point of $P(R;d_1,d_2)$ is a point of R. In [2], a digital region is defined to be convex if and only if it has the area property. Moreover, it is shown in [2,4] that the above three definitions are equivalent, the definition of Sklansky having been slightly modified.

In [11] Rosenfeld introduced the chord property to characterize digital straight lines. The chord property and the area property turned out to be equivalent [5]. Furthermore, it was shown in [3,4] that a digital region is convex if and only if any two points of the region may be connected by a digital straight line segment in the region.

These results lead us to believe that convexity of digital regions is now well defined and understood. Because of the recent growing interest in 3-dimensional image processing [1,6,9,16], it seems essential to develop a theory of 3-dimensional discrete geometry.

In this paper we introduce a definition of convexity of digital solids, which is an extension of that of Sklansky. Then we show that a digital solid is convex if and only if it has the chordal triangle property. However, this characterization of convexity does not lend itself to development of an efficient algorithm to recognize convex digital solids. Thus, convexity of digital solids is further characterized in terms of semi-

digital points on the surface of the convex hull of the digital solid. With this characterization, an efficient algorithm is developed to determine whether or not a given digital solid is convex.

The geometric properties that characterize convex digital regions are either well defined in or easily extendable to the 3-dimensional case. It will be shown that the chordal triangle property, which is an extension of the chord property, is the only one that is useful for characterizing the convexity of digital solids.

In the next section, we introduce a few relevant definitions, notation and terminology. Section 3 introduces a definition of convexity of digital solids and discusses its properties. An efficient algorithm for recognizing convex digital solids is presented in Section 4. The next section is concerned with geometric properties that characterize convex digital regions and solids.

2. Definitions

Consider the set of all lattice points in 3-dimensional Euclidean space. A lattic point $d = (h,k,m)$ is called a digital point. To each digital point d is associated a unit cube whose center is the digital point and faces are parallel to the coordinate planes. The cube associated with digital point d is denoted by c and is called a cell. Digital point d' is a 6-, 18-, or 26-neighbor of digital point d if c' shares with c a face, an edge or a corner point, respectively.

Let R be a set of digital points. Then $\overline{R}$ denotes its complement, that is, the set of all digital points that are not in R. We denote by R' the set of all cells that are associated with the points in R. The set of points cf 3-dimensional Euclidean space in R' is denoted by $s(R)$, and its boundary surface by $\partial s(R)$. For any two points x and y, $\overline{xy}$ denotes the line segment between the two.

A chain is a finite sequence of digital points such that every element of the sequence except the first is a 6-neighbor of its predecessor. A set R of digital points is said to be 6-connected if for any two points d,d' of R, there is a chain from d to d' in R.

Digital solids

A digital solid S is a finite set of digital points which is 6-connected. A digital solid S is said to be simple if

there exists no pair of points d_1, d_2 of S such that $\overline{d_1 d_2}$ is parallel to a coordinate axis and there is a point of $\bar{S}$ on $\overline{d_1 d_2}$. We denote by H(S) the convex hull of S, that is, H(S) is the smallest convex polyhedron that contains S. Note that the vertices of H(S) are points of S.

Digital image

A digital solid S is the digital image of a solid q, denoted S=I(q), if

(i) $q \subseteq s(S)$, and

(ii) if c is an element of S', then $c^{o} \cap q \neq \emptyset$, where c^{o} is the interior of c.

Half-cell expansion [15]

Let S be a digital solid and Q the set of all corner points of the cells of S'. Then Q is a digital solid, the points of Q being considered as digital points, and we denote Q by E(S), the half-cell expansion of S.

Digital convexity

A digital solid S is said to be digitally convex (or simply "convex") if it is simple and there is a convex solid q such that E(S)=I(q), that is, the half-cell expansion of S is the image of the convex solid q.

Chordal triangle property

Let w=(x,y,z) and w'=(x',y',z') be two points in 3-dimensional Euclidean space. The two points are said to be near each other

if $\max\{|x-x'|,|y-y'|,|z-z'|\}<1$. Note that no two digital points are near each other.

Let S be a digital solid. A point w is said to be near S if there is a point of S which w is near. Let d_1, d_2 and d_3 be points of S, which are not necessarily distinct. The triangle, possibly a degenerate one, whose vertices are d_1, d_2 and d_3, is called a chordal triangle of S. A chordal triangle T is said to lie near S if every point of T is near S. A digital solid S is said to have the chordal triangle property if every chordal triangle of S lies near it.

3. Digital convexity and the chordal triangle property

The main result of this section is that the chordal triangle property is a necessary and sufficient condition for a digital solid to be convex. First we show that a digital solid S has the chordal triangle property if and only if every point of H(S), the convex hull of S, is near S. Next it is shown that every point of H(S) is near S if and only if S is digitally convex. The main result then follows immediately.

Lemma 1: A digital solid S has the chordal triangle property if and only if every point of H(S) is near S.

Proof: Suppose that every point of H(S) is near S, and let T be a chordal triangle of S. Since T is a subset of H(S), every point of T is near S and T lies near S. Therefore, S has the chordal triangle property.

Now suppose that there are points of H(S) which are not near S, and let w be such a point. If w is on a face of H(S), then obviously w is a point of a chordal triangle. Hence, S does not have the chordal triangle property. Assume that w is an interior point of H(S). We further assume that w is not a digital point. (It turns out that the case where w is a digital point is taken care of when we consider the case where w is not a digital point.) Let c be the cell of which w is a point. Then the digital point d which is the center of c is not a point of S, since otherwise w is near S. There are two cases to consider.

Case 1: The digital point d is not a point of H(S).

Let u be the point at which $\overline{dw}$ intersects a face of H(S). We claim that u also is not near S. If so, u is a point on a face of H(S) and not near S. Thus, S does not have the chordal triangle property. It remains to prove our claim. If $d=(h,k,m)$, then $w=(h+\Delta x,\ k+\Delta y,\ m+\Delta z)$, where $|\Delta x|\le 1/2$, $|\Delta y|\le 1/2$ and $|\Delta z|\le 1/2$. Without loss of generality assume that $0\le\Delta x,\Delta y,\Delta z\le 1/2$. Then the point u is such that $u=(h+\Delta x',k+\Delta y',m+\Delta z')$ and $0\le\Delta x'\le\Delta x$, $0\le\Delta y'\le\Delta y$ and $0\le\Delta z'\le\Delta z$. (See Figure 1.) Suppose u is near S.

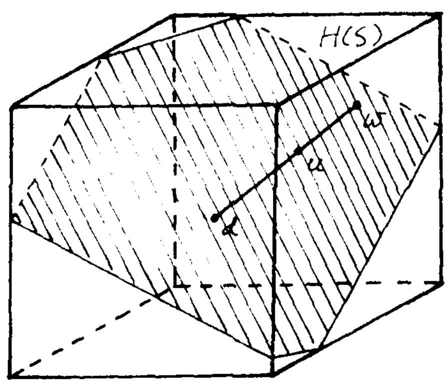

Figure 1.

Then there is a point d_1 in S which is near u. If $d_1=(h+\delta_x,k+\delta_y,m+\delta_z)$, then $\delta_t=1$ if $\Delta t>0$ and $\delta_t=0$ if $\Delta t=0$ for all t=x, y and z. Thus w is also near d_1, which is a contradiction. This proves our claim.

Case 2: The digital point d is a point of H(S).

Obviously, d is not near S (if it were, it would have to be in S, making w near S). As mentioned above, this also takes care of the case where w is a digital point (so that w=d).

Consider the set X of all digital points of H(S) that lie on the line ℓ_x which passes through d and is parallel to the x-axis. If X contains points of S on both sides of d, let u and v be the nearest points of S on each side of d. The line segment $\overline{uv}$ is a degenerate chordal triangle of S and it does not lie near S. Therefore, S does not have the chordal triangle property. Suppose that X does not contain any point of S on one side of d. Let u be the point at which ℓ_x intersects a face of H(S) and there is no point of S on $\overline{du}$. We claim that u is not near S and thus S does not have the chordal triangle property. Now we prove our claim. If u is a digital point, then obviously it is not near S. So assume that u is not a digital point. If d=(h,k,m), then u=(h+x,k,m) because u is a point on ℓ_x. Also x is not an integer, since u is not a digital point. The only digital points near u are d'=$(h+\lfloor x\rfloor,k,m)$ and d"=$(h+\lceil x\rceil,k,m)$, where $\lfloor x\rfloor$ is the largest integer not greater than x and $\lceil x\rceil$ the smallest integer not less than x. Neither d' nor d" is a point of S and u is not near S. So our claim is proved. This completes the proof of this lemma.$\sqcup$

We need a few lemmas before we derive a result on the relationship between the convex hull and convexity of a digital solid.

Lemma 2: Let S be a digital solid. If c is a cell of E(S), then $c\cap H(S)\neq\emptyset$.

Proof: If c is a cell of E(S), at least one corner point of c, say e, is a point of S. Then $e\in c\cap H(S)$. $\sqcup$

Lemma 3: Let S be a digital solid. If every point of H(S) is near S, then S is simple.

Proof: Suppose every point of H(S) is near S. Then every digital point of H(S) is a point of S. Consider any pair of points d_1, d_2 of S such that $\overline{d_1 d_2}$ is parallel to a coordinate axis. Since H(S) is convex, $\overline{d_1 d_2}$ is a subset of H(S). Therefore, every digital point on $\overline{d_1 d_2}$ is a point of S and S is simple by definition. $\square$

Lemma 4: Let S be a digital solid. If every point at H(S) is near S, then H(S) is a subset of the set of interior points of s(E(S)). Therefore, there exists a positive number ε such that $\mathrm{dist}(u,v) \geq \varepsilon$ for any point u of H(S) and any point v of $\partial s(E(S))$.

Proof: Let w be a point of $\partial s(E(S))$ and c be a cell of E(S) to which w belongs. If w is a corner point of c, then it is not near S. If w is on an edge of c, neither end point of the edge is a point of S; hence w is not near S. If w is on a face of c, no corner point of the face is a point of S, and again w is not near S. Therefore, H(S) and $\partial s(E(S))$ do not meet or intersect. Since every point of S is an interior point of s(E(S)), the vertices of H(S) are interior points of s(E(S)). Thus, H(S) is a subset of the set of interior points of s(E(S)).

Since $\partial s(E(S))$ and H(S) are both compact sets and do not have any point in common, $\min_{u \in H(S),\, v \in \partial s(E(S))} \mathrm{dist}(u,v) = \varepsilon > 0$. $\square$

Lemma 5: Let S be a digital solid. If q is a convex solid such that $E(S) \subseteq I(q)$, then $H(S) \subseteq q^o$, where q^o is the interior of q.

Proof: Let v be a vertex of H(S). Then there are eight cells of E(S) of which v is a corner point. Since H(S) is convex, there is one cell among the eight, say c, such that $c^o \cap H(S) = \emptyset$. Therefore, $s(E(S)) \not\subseteq I(H(S))$. Since obviously $H(S) \subseteq q$, H(S) must be a proper subset of q. Suppose $r = \partial q \cap \partial H(S)$, the intersection of the boundary surfaces of q and H(S), is not empty. Since q is convex, r contains a vertex of H(S), say v, and there is a plane p such that v is on p and q lies in one side of p. Then there is a cell of E(S), call it c, such that v is a corner point of c and c lies on the opposite side of p from q. Thus, $c^o \cap q = \emptyset$ and $E(S) \not\subseteq I(q)$, which is a contradiction. Therefore, r is empty and $H(S) \subseteq q^o$. $\square$

Lemma 6: A digital solid S is convex if and only if every point of H(S) is near S.

Proof: Suppose that every point of H(S) is near S. Let q be a convex polyhedron obtained from H(S) by parallel translation of all its faces outward by a distance δ. By Lemma 4, there exists a positive number δ, which depends on ε in the lemma, such that q is a subset of s(E(S)). If c is a cell of E(S), $c \cap H(S) \neq \emptyset$ by Lemma 2 and thus $c^o \cap q \neq \emptyset$ since q is an expansion of H(S). Hence, $E(S) = I(q)$. Also by Lemma 3, S is simple. Therefor, S is digitally convex.

Now suppose that there are points of $H(S)$ which are not near S. Since every interior point of $s(E(S))$ is near S, $H(S)$ contains a point w which is either a point of $\partial s(E(S))$ or of $\overline{s(E(S))}$. Thus, w is a point of a cell c which is not a cell of $E(S)$. Suppose there exists a convex solid q whose image is $E(S)$. Since $E(S) \subseteq I(q)$, $H(S) \subseteq q^{o}$ by Lemma 5. Thus w must be an interior point of q and $c^{o} \cap q \neq \emptyset$. Then $c \subset I(q)$ but c is not a cell of $E(S)$, which is a contradiction. Therefore, S is not digitally convex because there is no convex solid whose image is $E(S)$. $\Box$

The main result of this section follows immediately from Lemmas 1 and 6:

Theorem 7: A digital solid is convex if and only if it has the chordal triangle property.

4. Algorithm for recognition of convex digital solids

In the previous section we showed that the chordal triangle property is a simple geometric property that characterizes convex digital solids. However, it does not lend itself to development of an efficient algorithm to determine whether or not a digital solid is convex. The reason is that there are infinitely many points on a chordal triangle and the number of chordal triangles is $O(n^3)$, where n is the number of points in S. In the sequel, it will be shown that only finitely many points on the surface of H(S) need be examined to recognize the convexity of a digital solid.

Let p be a polygon on a plane in 3-dimensional Euclidean space. A point $w=(x,y,z)$ of p is said to be a semi-digital point if at least two of its coordinates are integers or it is on an edge of p and one of its coordinates is an integer. Note that the number of semi-digital points on p is approximately equal to its area.

Our algorithm is based on the result stated in the following theorem.

Theorem 8: A simple digital solid S is convex if and only if every semi-digital point on the faces of H(S) is near S.

Proof: "Only if" is a special case of Lemma 6. Conversely, suppose that the simple digital solid S is not convex. By Lemma 6, there are points of H(S) which are not near S; let

$w=(x,y,z)$ be such a point. There are two cases to consider.

Case 1: $w=(x,y,z)$ is a digital point, i.e., x,y,z are all integers.

Let ℓ_x be the ray from w parallel to the x-axis; then ℓ_x contains no point of S because S is simple. Let u be the point at which ℓ_x intersects a face of $H(S)$. Then u is not near S and $u=(x',y,z)$ is a semi-digital point.

Case 2: $w=(x,y,z)$ is not a digital point.

Let c be the cell of which w is a point. Then d, the center of c, is not a point of S, since otherwise w is near S. If d is a point of $H(S)$, then Case 1 applies to it, and there is a semi-digital point on a face of $H(S)$. Thus, assume that d is not a point of $H(S)$. Let u be the point at which $\overline{dw}$ intersects a face of $H(S)$ (see Figure 1.). Then u is not near S. (Refer to the proof of Lemma 1 for the reason.) Let $d=(h,k,m)$. Then $u=(h+\Delta x,k+\Delta y,m+\Delta z)$, where $|\Delta x|\leq 1/2, |\Delta y|\leq 1/2$ and $|\Delta z|\leq 1/2$.

(a) u is on an edge of $H(S)$.

As we move u along the edge in either direction, $\Delta x, \Delta y$ and Δz change their values continuously and monotonically. Let us move u until one of $\Delta x, \Delta y$ and Δz attains a value of 0, 1, or -1 for the first time. This always occurs because the endpoints of the edge are digital points. The resulting u is a semi-digital point. It is easy to see that u is not near S.

(b) u is not on an edge of $H(S)$.

Take a coordinate plane which is not parallel to the face, say the xy-plane. Consider the line ℓ_{xy} of intersection

of the face and the plane which is parallel to the xy-plane and contains the point u. As the point u is moved along ℓ_{xy}, the values of Δx and Δy change while that of Δz is kept constant. We move u until either it reaches an edge of the face or one of Δx and Δy attains a value of 0, 1, or -1 for the first time. If u reaches an edge first, then we have Case 2(a), which has already been treated. Now assume without loss of generality that Δx attains a value of 0, 1, or -1 first. Then the x-coordinate of u is an integer. Consider the line ℓ_{yz} of intersection of the face and the plane which is parallel to the yz-plane and contains u. (If the yz-plane were parallel to the face, then Δy would have attained a value of 0, 1, or -1 first, since Δx would have stayed constant. In this case ℓ_{xz} would be considered instead.) Move the point u along ℓ_{yz} until either it reaches an edge of the face or the value of Δy or Δz becomes 0, 1, or -1 for the first time. If u reaches an edge first, then it is a semi-digital point. If Δy or Δz attains a value of 0, 1 or -1 first, then again u is a semi-digital point, since two of its coordinates are integers. It can easily be shown that u is not near S. □

Corollary 9: A digital solid S is convex if and only if it is simple and every semi-digital point on the surface of H(S) is near S.

We are now ready to present an algorithm to determine whether or not a given digital solid is convex. In fact, Corollary 9

concisely describes the algorithm. The algorithm first checks if S is or is not simple. If S is not simple, then the algorithm outputs that S is not convex and halts. Otherwise it examines every semi-digital point on the surface of H(S) to see whether any one of them is not near S. If such a point is found, the algorithm halts after it outputs that S is not convex. If all semi-digital points are near S, then S is convex. Below the algorithm is presented formally.

Algorithm 3D-CONVEXITY(S)

1. If S is not simple then output (False); stop.
2. Construct the convex hull H(S) of S.
3. If a semi-digital point which is not near S is found on the surface of H(S) then output (False); stop.
4. Output (True); stop.

The correctness of the algorithm is immediate from Corollary 9. However, determination of its computational complexity requires a detailed description of each step and the data structures used in the algorithm. For simplicity we assume that a digital solid S is a subset of the set of n^3 digital points in the cube whose edge is of length n. S is represented by a run length code [12] such that RC(i,j) is a finite sequence of run legnths of 0's, digital points of $\bar{S}$ with coordinates (i,j,z), and 1's, digital points of S with coordinates (i,j,z). Thus, $RC(i,j)=(\ell_{ij0},\ell_{ij1},\ldots,\ell_{ijr_{ij}})$ represents the (i,j)th row as composed of a run of 0's of length ℓ_{ij0} followed by a run of 1's of length ℓ_{ij1} and so on.

(1) Is S simple?

If a row has more than one run of 1's, then S is obviously not simple. Consider the set of rows in a plane parallel to the yz-plane, that is, the set of rows (i,j) for all j, $1 \le j \le n$, and a fixed i, $1 \le i \le n$. Suppose that $\ell_{ij0} < \ell_{i,j+1,0}$ and $\ell_{ik0} > \ell_{i,k+1,0}$ for some $1 \le j < k \le n$. Then S is not simple. Also, if $\ell_{ij0} + \ell_{ij1} > \ell_{i,j+1,0} + \ell_{i,j+1,1}$ and $\ell_{ik0} + \ell_{ik1} < \ell_{i,k+1,0} + \ell_{i,k+1,1}$, then S is not simple. The set of rows in a plane parallel to the xz-plane can be checked similarly. It is easy to see that S is simple otherwise. The above observations lead to algorithm SIMPLE for step 1 of algorithm 3D-CONVEXITY.

<u>Algorithm</u> SIMPLE(S)

1.1. <u>For</u> each i and each j, $1 \le i,j \le n$,

check RC(i,j) to see if there is more than one run of 1's; if so, output (<u>False</u>); stop.

1.2. <u>For</u> each i, $1 \le i \le n$,

check if ℓ_{ij0} increases and then decreases or $\ell_{ij0} + \ell_{ij1}$ decreases and then increases as j increases from 1 to n; if so, output (<u>False</u>); stop.

1.3. <u>For</u> each j, $1 \le j \le n$,

check if ℓ_{ij0} increases and then decreases or $\ell_{ij0} + \ell_{ij1}$ decreases and then increases as i increases from 1 to n; if so, output (<u>False</u>); stop.

Each step runs in $O(n^2)$ time and requires a constant work space.

(2) Construction of H(S)

A point of S is a corner point if three of its 6-neighbors are points of $\overline{S}$ and they are mutually 18-neighbors. Only a corner point of S may be a vertex of H(S), and the convex hull of the set of corner points of S is also the convex hull of S. Obviously, only the first and the last digital points of each row can be corner points. Hence, there are at most $O(n^2)$ corner points in S because there are n^2 rows. Let CP(S) denote the set of corner points of S.

An algorithm to build the convex hull of a set of points is given in [8]. We denote the algorithm by HULL. Its input is a set of points R and its output is the convex hull $H(R)=(F_1,\ldots,F_k)$, where each face F_i of H(R) is represented by a sequence of its vertices.

Algorithm CONVEXHULL(S,H(S),k)

2.1. Obtain CP(S), the set of corner points of S, by checking the first and last points of S in each row (i,j), where $1\le i,j\le n$.

2.2. Call HULL(CP(S),H(S)) to construct $H(S)=(F_1,\ldots,F_k)$.

2.3. Return.

Step 2.1 requires $O(n^2)$ computing time and work space. Step 2.2 runs in $O(n^2 \log n)$ time and needs $O(n^2)$ work space to store H(S).

(3) Is every semi-digital point near S?

Since the number of semi-digital points is approximately equal to the area of the surface of H(S) and H(S) is convex, there are $O(n^2)$ semi-digital points. Given a face of H(S) represented by a sequence of its vertices, each semi-digital point may be located in constant computing time. If a semi-digital point is a digital point, then it is near S if and only if it is a point of S. If a semi-digital point has two integer coordinates, then two digital points must be checked. For instance, if a semi-digital point is $w=(h+\Delta x,k,m)$, $0<\Delta x<1$, then it is near S if either (h,k,m) or (h+1,k,m) is a point of S. If a semi-digital point w has only one integer coordinate, then four digital points must be checked to see if w is near S. Thus, we have the following algorithm for step 3 of algorithm 3D-CONVEXITY.

Algorithm NEAR(S,H(S),k)

3.1. For each face F_i of H(S), $1 \le i \le k$,

for each semi-digital point w on F_i

check if w is near S; if not, output (False); stop.

3.2. Return.

Obviously algorithm NEAR has running time of $O(n^2)$ and needs constant work space.

Theorem 10: Algorithm 3D-CONVEXITY determines whether or not a digital solid is convex, runs in $O(n^2 \log n)$ time and requires $O(n^2)$ work space.

5. Convex digital regions and solids

We restate formal definitions of the geometric properties that are used to define convexity of digital regions.

Line property

A digital region R is said to have the line property if there is no triplet (d_1,d_2,d_3) of collinear digital points such that d_1 and d_3 are points of R and d_2 is a point of $\overline{R}$.

Area property

Let $\partial s(R)$ denote the boundary of the set of points of cells whose centers are the points of R. R is said to have the area property if there are no two points d_1,d_2 of R such that the bounded area whose boundaries consist of nonempty segments of $\overline{d_1d_2}$ and $\partial s(R)$ contains a digital point of $\overline{R}$.

Chord property

A digital region R is said to have the chord property if for each point $w=(x,y)$ on $\overline{d_1d_2}$, where d_1 and d_2 are any two points of R, there is a point $d=(h,k)$ of R such that $\max\{|x-h|,|y-k|\}<1$.

All three properties above are equivalent in that a digital region is convex if and only if it has any one of them. There are Euclidean geometric properties of which the line property and the area property may be considered digital equivalents, but this is not the case for the chord property. It is obvious that the line and chord properties are well defined in 3-dimensional discrete geometry. It is immediate that the chordal

triangle property is an extension of the chord property, but there is no natural extension of the line property. In [2] it was shown that a digital region R has the area property if and only if the convex hull of R contains no point of $\overline{R}$.
Thus, the volume property which is defined below may be considered as an extension of the area property.

Volume property

A digital solid S is said to have the volume property if H(S) contains no point of $\overline{S}$.

In Section 3, we proved that a digital solid is convex if and only if it has the chordal triangle property. In the sequel, we show that every other property mentioned here is a necessary but not a sufficient condition for a digital solid to be convex.

Theorem 11: Each one of the line, chord, and volume properties is a necessary but not a sufficient condition for a digital solid to be convex.

Proof: Suppose that a digital solid S does not have either the line property or the chord property, then trivially S does not have the chordal triangle property. Hence, S is not convex by Theorem 7. Now suppose that S does not satisfy the volume property. Then not every point of H(S) is near S and S is not convex by Lemma 6.

To see that these conditions are not sufficient, consider the polyhedron q which is formed by the three coordinate planes

and two planes whose equations are $x/3+y/3=1$ and $x/3+y/6+z/2=1$, respectively. Let S_1 be the digital solid that consists of all digital points of q except the point $d=(1,1,1)$. (See Figure 2.)

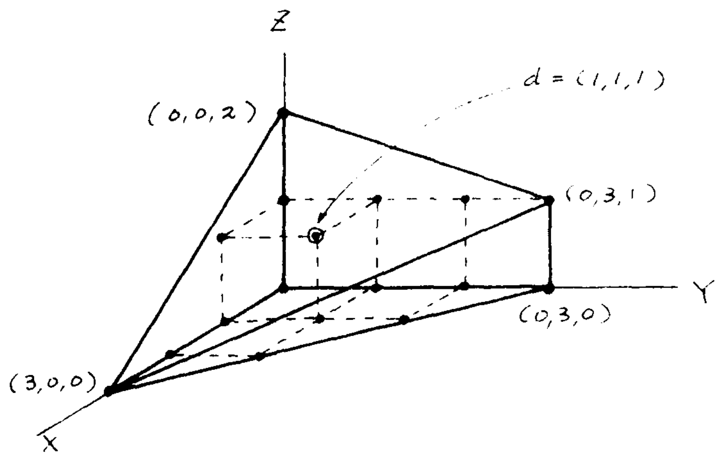

Figure 2. A digital solid which is not convex but has both the line property and the chord property.

Then q is $H(S_1)$, the convex hull of S_1. Since it is not a point of S but is a digital point, d is a point of $H(S_1)$ which is not near S_1. Thus, S_1 is not convex. Note that every point of $H(S_1)$ except d is near S_1. It is easy to see that there are no two points d_1, d_2 of S such that d is on $\overline{d_1 d_2}$. Therefore, S_1 has both the line property and the chord property.

Next let S_2 be the digital solid $\{(0,0,0),(0,0,1),(0,0,2),(0,1,2),(1,1,2),(1,0,0)\}$. (See Figure 3.) Then the point $w=(1,1/2,1)$ is a point of $H(S_2)$ and not near S_2, so S_2 is not convex. Since $H(S_2)$ contains no point of $\bar{S}_2$, S_2 has the volume property. $\square$

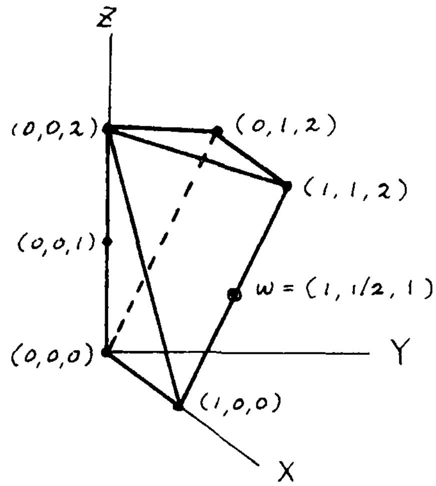

Figure 3. A digital solid which is not convex but has the volume property.

6. Conclusion

A definition of convexity of digital solids was introduced. It is a 3-dimensional extension of the modified definition of Sklansky [3,13]. Just as convexity of digital regions is characterized by the chord property, convexity of digital solids is characterized by the chordal triangle property, which is a 3-dimensional extension of the chord property. The main question concerning convexity of digital solids is under what conditions a digital solid is the digitization of a convex solid. The definition presented in this paper and the characterization of convexity by the chordal triangle property seem to provide a satisfactory answer to the question.

There are other geometric properties that are used to characterize convexity of digital regions. Somewhat surprisingly, they or their 3-dimensional extensions turn out to be only necessary conditions. It would be interesting to determine the class of digital solids that each of these properties characterizes.

If, given a digital solid, a sequential algorithm must construct its convex hull to determine its convexity, the time complexity obtained here cannot be improved. For, O(k log k) is the optimal time complexity for any sequential algorithm to construct the convex hull of k points. Thus, to develop a faster algorithm, convexity of digital solids must be characterized by a simpler geometric property.

References

1. G. T. Herman and H. K. Liu, "Dynamic boundary surface detection," CGIP 7 (1978), 130-138.

2. C. E. Kim, "On the cellular convexity of complexes," TR-875 (1980), Department of Computer Science, University of Maryland.

3. C. E. Kim and A. Rosenfeld, "Digital straight lines and convexity of digital regions," TR-876 (1980), Department of Computer Science, University of Maryland.

4. C. E. Kim and J. Sklansky, "Digital convexity," in preparation.

5. C. E. Kim, "On cellular straight line segments," TR-918 (1980), Department of Computer Science, University of Maryland.

6. H. K. Liu, "Two- and three-dimensional boundary detection," CGIP 6 (1977), 123-134.

7. M. Minsky and S. Papert, Perceptrons, MIT Press, 1969.

8. F. P. Preparata and S. J. Hong, "Convex hulls of finite sets of points in two and three dimensions," CACM 20 (1977), 87-93.

9. D. R. Reddy and S. Rubin, "Representation of three-dimensional objects," CMU-CS-78-113 (1978).

10. A. Rosenfeld, "Connectivity in digital pictures," JACM 17 (1970), 146-160.

11. A. Rosenfeld, "Digital straight line segments," IEEE Trans. on Comput. C-23 (1974), 1264-1269.

12. A. Rosenfeld and A. C. Kak, Digital Picture Processing, Academic Press, NY, 1976.

13. J. Sklansky, "Recognition of convex blobs," Pattern Recognition 2 (1970), 3-10.

14. J. Sklansky, R. L. Chazin and B. J. Hansen, "Minimum-perimeter polygons of digitized silhouettes," IEEE Trans. on Comput. C-21 (1972), 250-268.

15. J. Sklansky, "Measuring concavity on a rectangular mosaic," IEEE Trans. on Comput. C-21 (1972), 1355-1364.

16. S. W. Zucker and R. A. Hummel, "An optimal three-dimensional edge operator," Proc. PRIP (1979), 162-168.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) CONVEX DIGITAL SOLIDS		5. TYPE OF REPORT & PERIOD COVERED Technical
		6. PERFORMING ORG. REPORT NUMBER TR-929
7. AUTHOR(s) Chul E. Kim		8. CONTRACT OR GRANT NUMBER(s) AFOSR-77-3271
9. PERFORMING ORGANIZATION NAME AND ADDRESS Computer Vision Laboratory, Computer Science Center, University of Maryland, College Park, MD 20742		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Math. & Info. Sciences, AFOSR/NM Bolling AFB Wash., DC 20334		12. REPORT DATE August 1980
		13. NUMBER OF PAGES 27
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Image processing
Pattern recognition
Shape
Convexity
Three-dimensional arrays

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A definition of convexity of digital solids is introduced. Then it is proved that a digital solid is convex if and only if it has the chordal triangle property. Other geometric properties which characterize convex digital regions are shown to be only necessary, but not sufficient, conditions for a digital solid to be convex. An efficient algorithm is presented that determines whether or not a digital solid is convex.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)